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# Continuous & Piecewise Concave Behavior Of Maximum & Minimum Values Of Some Generalized Fuzzy Entropy

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## Abstract

This research paper proposes a new method of examining the continuous and piecewise concave behavior of maximum and minimum values of some well known existing measures of generalized fuzzy entropy subject to the total fuzziness. It has been proved that the maximum value is a continuous and concave function whereas the minimum value is continuous and piecewise concave function which vanishes for every known integral values. It is the fuzzy set which can describe fuzzy objects effectively, play an important role in system modeling, system designing, fuzzy pattern recognition system, fuzzy control system, fuzzy management information system, fuzzy knowledge based system, fuzzy decision making, fuzzy neural networking etc and thus making the quantitative analysis of fuzziness in a fuzzy set an important problem.

**Keywords:** Fuzzy vector, Fuzzy entropy, piecewise concavity, maximum and minimum values.

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## Introduction

Fuzzy set theory developed by Zadeh[10] received recognition from different quarters and after its introduction, a considerable body of literature blossomed around this concept. Fuzzy sets theory makes use of entropy- an important concept in information theory, to measure the degree of fuzziness in a fuzzy set, which is called fuzzy entropy and thus has especial important position in various fuzzy systems. It is the fuzzy set which can describe fuzzy objects effectively, play an important role in system modeling, system designing, fuzzy pattern recognition system, fuzzy control system, fuzzy management information system, fuzzy knowledge based system, fuzzy decision making, fuzzy neural networking etc and thus making the quantitative analysis of fuzziness in a fuzzy set an important problem. For example, when generalized fuzzy entropy is used as learning criterion for neural networks, efficient structure parameters are obtained quickly. Jayne's[4] Maximum Entropy Principle aims at maximizing uncertainty subject to given constraints and it restricts its use only to Shannon's[9] uncertainty measure. Uncertainty is much deeper concept to be captured by probability theory alone and probabilistic entropy is too deeper to be captured by Shannon's[9] measure of entropy alone. Thus, there is need for generalized measures of entropy just to extend the scope of their applications for the study of different optimization principles. These generalized measures of maximum entropy require us to choose that distribution, which maximizes a specified measure of entropy. A large number of applications of Maximum Entropy Principle in Science and Engineering have been provided by Kapur and Kesavan [7], Kapur et al.[6] whereas some optimization principles towards crop area have been discussed and investigated by Hooda and Kapur[3].

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**1.1.Purpose of the present paper**. Consider a given fuzzy set $A$ with $n$ supporting points $(x_1, x_2, ......x_n)$ corresponding to the fuzzy vector $(\mu_A(x_1), \mu_A(x_2), ......\mu_A(x_n))$ where $\mu_A(x_i)$ is the degree of membership of the elements $x_i$ of the set .Our purpose is to find the maximum and minimum values of some generalized measures of fuzzy entropy subject to the total fuzziness $\sum_{i=1}^{n} \mu_A(x_i) = k,\ 0 \le k \le n$ and to examine the continuous and piecewise concave behavior of these values. For this, we have considered some well known existing generalized measures of fuzzy entropy which are given below.

Corresponding to Renyi's [8] probabilistic entropy, Bhandari and Par[1]gave the following generalized measure of fuzzy entropy :

$$H_\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right], \alpha \ne 1, \alpha > 0 \qquad ...(1.1)$$

Corresponding to Havrda and Charvat's[2] probabilistic entropy, Kapur[5] gave the following generalized measure of fuzzy entropy :

$$H^\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1\right],\ \alpha \ne 1, \alpha > 0 \qquad ...(1.2)$$

Kapur's[5]measure of generalized fuzzy entropy of order $\alpha$ and type $\beta$ is given by the following mathematically expression:

$$H_{\alpha,\beta}(A) = \frac{1}{\beta-\alpha} \log\left[\frac{\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right]}{\sum_{i=1}^{n}\left[\mu_A^\beta(x_i) + (1-\mu_A(x_i))^\beta\right]}\right], \alpha \ge 1, \beta \le 1 \text{ or } \alpha \le 1, \beta \ge 1, \alpha \ne \beta \ ...(1.3)$$

It has been proved that the maximum values of the generalized fuzzy entropy given by (1.1),(1.2) and (1.3) subject to the total fuzziness $\sum_{i=1}^{n} \mu_A(x_i) = k,\ 0 \le k \le n$ are continuous and concave function of $k$ while the minimum value is a continuous and piecewise concave function of $k$ which vanishes for every positive integer $k$ and has maximum value for any fractional value of $k$.

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**2.1 Maximum Value of $H_\alpha(A)$ :** Take $f(x)=x^\alpha+(1-x)^\alpha$, $0\le x\le 1$; $\alpha\ne 1,\alpha>0$

Differentiating it w.r.t. $x$, $f'(x)=\alpha\left[x^{\alpha-1}-(1-x)^{\alpha-1}\right]$; $f''(x)=\left(\alpha^2-\alpha\right)\left[x^{\alpha-2}+(1-x)^{\alpha-2}\right]$

Consider the following cases.

**Case I.** For $\alpha>1$, $f''(x)>0 \Rightarrow \log f''(x)>0 \Rightarrow \frac{1}{1-\alpha}\log f''(x)<0\ \forall\ \alpha>1$

$\frac{1}{1-\alpha}\log f(x)$ is a concave function of $x$ for all $\alpha>1$. Using the fact that sum of concave functions is again a concave function, therefore, for $H_\alpha(A)$ is a concave function of $\mu_A(x_i)$ for each $\alpha>1$.

**Case II.** For $0<\alpha<1$, $f''(x)<0 \Rightarrow \log f''(x)<0 \Rightarrow \frac{1}{1-\alpha}\log f''(x)<0\ \forall\ \alpha<1$. $\therefore \frac{1}{1-\alpha}\log f(x)$ is a concave function of $\alpha$ for all $0<\alpha<1$. Hence, $H_\alpha(A)$ is a concave function of $\mu_A(x_i)$ for all $0<\alpha<1$.It implies that $H_\alpha(A)$ is a concave function of $\mu_A(x_i)$ for each $\alpha$. Therefore, its maximum value exists. For maximum value, we put

$\frac{\partial H_\alpha(A)}{\partial \mu_A(x_i)}=0$, which gives $\frac{\alpha}{\alpha-1}\left[\frac{\mu_A^{\alpha-1}(x_i)-\left(1-\mu_A(x_i)\right)^{\alpha-1}}{\mu_A^{\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}}\right]=0$

$$\Rightarrow \quad \mu_A^{\alpha-1}(x_i)-\left(1-\mu_A(x_i)\right)^{\alpha-1}=0 \Rightarrow \mu_A(x_i)=1-\mu_A(x_i) \Rightarrow \mu_A(x_i)=\frac{1}{2}$$

Also $\sum_{i=1}^{n}\mu_A(x_i)=k$ gives $\frac{n}{2}=k$ or $\frac{k}{n}=\frac{1}{2}=\mu_A(x_i)$.Therefore, the maximum value of $H_\alpha(A)$ occurs at $k=\frac{n}{2}$ and the maximum value is given as :

$$Max.H_\alpha(A)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\left(\frac{k}{n}\right)^\alpha+\left(1-\frac{k}{n}\right)^\alpha\right]=\frac{n}{1-\alpha}\log\left[\frac{k^\alpha+(n-k)^\alpha}{n^\alpha}\right]$$
...(2.1)

$$=\frac{n}{1-\alpha}\log g(k), \text{ where } \ g(k)=\frac{k^{\alpha}+(n-k)^{\alpha}}{n^{\alpha}} \qquad ..(2.2)$$

Differentiating equation (2.2) w.r.t. $k$, we get

$$g'(k)=\alpha\left[\frac{k^{\alpha-1}-(n-k)^{\alpha-1}}{n^{\alpha}}\right]; g''(k)=(\alpha^2-\alpha)\left[\frac{k^{\alpha-2}+(n-k)^{\alpha-2}}{n^{\alpha}}\right]$$

(i) When $\alpha>1$ then $g''(k)>0 \Rightarrow \log g''(k)>0 \Rightarrow \frac{n}{1-\alpha}\log g''(k)<0 \quad \forall \alpha>1$

$\therefore$ $\frac{n}{1-\alpha}\log g(k)$ is a concave function of $k$ for all $\alpha>1$

(ii) When $0<\alpha<1$, then $g''(k)<0 \Rightarrow \log g''(k)<0 \Rightarrow \frac{n}{1-\alpha}\log g''(k)<0 \ \forall 0<\alpha<1$

$\therefore$ $\frac{n}{1-\alpha}\log g(k)$ is a concave function of $k$ for each $0<\alpha<1$.

Thus, for each $\alpha$, $\frac{n}{1-\alpha}\log g(k)$ is a concave function of $k$. Hence, the maximum value of $H_\alpha(A)$ is a concave function of $k$ for each $\alpha$, therefore, its maximum value occurs. For maximum value, we put

$\frac{d}{dk}(Max.H_\alpha(A))=0$, which gives

$$\frac{n\alpha}{1-\alpha}\left[\frac{k^{\alpha-1}-(n-k)^{\alpha-1}}{k^{\alpha}+(n-k)^{\alpha}}\right]=0 \quad \Rightarrow k^{\alpha-1}-(n-k)^{\alpha-1}=0 \Rightarrow k=n-k \Rightarrow \frac{k}{n}=\frac{1}{2}$$

Thus, the maximum value of $Max.H_\alpha(A)$ exists at $k=\frac{n}{2}$ and the maximum value is given as :

$$Max.\left[Max.(H_\alpha(A))\right]=\frac{n}{1-\alpha}\log\left[\left(\frac{1}{2}\right)^{\alpha}+\left(\frac{1}{2}\right)^{\alpha}\right] = \frac{n}{1-\alpha}\log 2^{1-\alpha}=n\log 2$$

Further, when

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(i) $k=0,\ Max.H_\alpha(A)=0$ and $\frac{d}{dk}\left(Max.H_\alpha(A)\right)=\frac{\alpha}{\alpha-1}\begin{cases}>0, & \alpha>1\\ <0, & 0<\alpha<1\end{cases}$

(ii) $k=\frac{n}{2}, Max.H_\alpha(A)=n\log 2$ and $\frac{d}{dk}\left(Max.H_\alpha(A)\right)=0$

(iii) $k=n, Max.H_\alpha(A)=0$ and $\frac{d}{dk}\left(Max.H_\alpha(A)\right) = \frac{\alpha}{1-\alpha}\begin{cases}<0, & \alpha>1\\ >0, & 0<\alpha<1\end{cases}$

Thus, we can say that the maximum value of $Max.H_\alpha(A)$ increases from 0 to $n\log 2$ as $k$ increases from 0 to $\frac{n}{2}$ and it decreases from $n\log 2$ to 0 as $k$ further increases from $\frac{n}{2}$ to $n$.

**3.1 Minimum value of** $H_\alpha(A)$**:**Consider the following cases.

**Case I.** When *k* is any positive integer, say $k=m$, then we can choose *m* values of $\mu_A(x_i)$ as unity

and other $(n-m)$ values of $\mu_A(x_i)$ as zero. i.e. , $\mu_A(x_i)=\{1,1,1,......1,0,0,......0\}$.

In this case, the minimum value of $H_\alpha(A)$ can be obtained as below. Equation (1.1), can be written as

$$H_\alpha(A)=\frac{1}{1-\alpha}\left[\sum_{i=1}^{m}\log\left[\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right]+\frac{1}{1-\alpha}\sum_{i=m+1}^{n}\log\left[\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha\right]\right]$$

$$\therefore \qquad Min.H_\alpha(A)=\frac{1}{1-\alpha}\left(n\log 1+n\log 1\right)=0$$

**Case II.** When *k* is any fraction, then we can write $k=m+\xi$, where *m* is any non-negative integer and

$\xi$ is a positive fraction. Then, we can choose *m* values of $\mu_A(x_i)$ as unity $(m+1)^{th}$ value as $\xi$ and

remaining $(n-m-1)$ values of $\mu_A(x_i)$ as zero i.e., $\mu_A(x_i)=(1,1,1,......1,\xi,0,0,......0)$.

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In this case, the minimum value of $H_\alpha(A)$ can be obtained as below. Equation (1.1) can be written as

$$H_\alpha(A)=\frac{1}{1-\alpha}\left[\begin{array}{r}\sum_{i=1}^{m}\log\left[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right]+\log\left[\mu_A^\alpha(x_{m+1})+(1-\mu_A(x_{m+1}))^\alpha\right]\\ +\sum_{i=m+2}^{n}\log\left[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right]\end{array}\right]$$

$\therefore\ Min.H_\alpha(A)=\frac{1}{1-\alpha}\log\left[\xi^\alpha+(1-\xi)^\alpha\right]=\frac{1}{1-\alpha}\log\phi(\xi)$, where ...(3.1)

$$\phi(\xi)=\xi^\alpha+(1-\xi)^\alpha;\quad \phi'(\xi)=\alpha\left[\xi^{\alpha-1}-(1-\xi)^{\alpha-1}\right];\quad \phi''(\xi)=(\alpha^2-\alpha)\left[\xi^{\alpha-2}+(1-\xi)^{\alpha-2}\right]$$

**Sub-Case I.**When $\alpha>1$, then $\phi''(\xi)>0\ \Rightarrow\ \phi(\xi)$ is a convex function of $\xi\Rightarrow\log\phi(\xi)$ is a convex function of $\xi\ \Rightarrow\ \frac{1}{1-\alpha}\log\phi(\xi)$ is a concave function of $\xi$ when $\alpha>1$.Therefore, $Min.H_\alpha(A)$ is a concave function of $\xi$ for each $\alpha>1$.

**Sub-Case II.**When $0<\alpha<1$, then $\phi''(\xi)<0\ \Rightarrow\ \phi(\xi)$ is a concave function of $\xi\Rightarrow\log\phi(\xi)$ is a concave function of $\xi\Rightarrow\frac{1}{1-\alpha}\log\phi(\xi)$ is a concave function of $\xi$ when $0<\alpha<1$.Therefore, $Min.H_\alpha(A)$ is a concave function of $\xi$ for each $0<\alpha<1$.

Hence, we can say that $Min.H_\alpha(A)$ is a concave function of $\xi$ for each $\alpha$. Hence, its maximum value occurs. For maximum value, we put

$$\frac{d}{d\xi}(Min.H_\alpha(A))=0\Rightarrow\frac{\alpha}{1-\alpha}\left[\frac{\xi^{\alpha-1}-(1-\xi)^{\alpha-1}}{\xi^\alpha+(1-\xi)^\alpha}\right]=0\Rightarrow\xi^{\alpha-1}-(1-\xi)^\alpha=0\Rightarrow\xi=1-\xi\ \Rightarrow\ \xi=\frac{1}{2}$$

Hence, the maximum value of $Min.H_\alpha(A)$ exists at $\xi=\frac{1}{2}$ and from equation (6), the maximum value is given as:

$Max.\ Min.(H_\alpha(A))=\frac{1}{1-\alpha}\log\left[\left(\frac{1}{2}\right)^\alpha+\left(\frac{1}{2}\right)^\alpha\right]=\frac{1}{1-\alpha}\log 2^{1-\alpha}=\log 2$.Further, when

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(i) $\xi = 0,\ Min.H_{\alpha}(A) = 0$

(ii) $\xi = \frac{1}{2},\ Min.H_{\alpha}(A) = \log 2$

(iii) $\xi = 1,\ Min.H_{\alpha}(A) = 0$

As $\xi$ increases from $\xi = 0$ to $\xi = 1/2$, the value of $Min.H_{\alpha}(A)$ increases from 0 to $\log 2$ and when $\xi$ further increases from $\xi = 1/2$ to $\xi = 1$, the value of $Min.H_{\alpha}(A)$ decreases from $\log 2$ to 0.

Thus, we can say that $Min.H_{\alpha}(A)$ is a concave function of $k$ when $k$ is any non-negative fractional value. Thus, we conclude that $Min.H_{\alpha}(A)$ is a piecewise concave function of $k$ which vanishes for every positive integer $k$ and has a maximum value of $\log 2$ for every positive fractional value of $k$.

**Conclusion**

Uncertainty is much deeper concept to be captured by probability theory alone and probabilistic entropy is too deeper to be captured by Shannon's[9] measure of entropy alone. Thus, there is need for generalized measures of entropy just to extend the scope of their applications for the study of different optimization principles. These generalized measures of maximum entropy require us to choose that distribution, which maximizes a specified measure of entropy. A large number of applications of Maximum Entropy Principle in Science and Engineering have been provided by Kapur and Kesavan [7], Kapur et al.[6] whereas some optimization principles towards crop area have been discussed and investigated by Hooda and Kapur[3].

It is concluded that that the maximum value of the generalized fuzzy entropy subject to the total fuzziness $\sum_{i=1}^{n} \mu_A(x_i) = k$ is a continuous and concave function of $k$ while the minimum value is continuous and piecewise concave function of $k$ which vanishes for every positive value of *k* and has a maximum value for any non-negative fractional value of k.

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